

Scattering of massive scalars by Schwarzschild black holes

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Abstract

The Klein-Gordon equation for the wave function of a single massive scalar is written in spherical and parabolic coordinates in the presence of a Schwarzschild background, and some semi-classical techniques for deriving asymptotic results are considered. In addition to the well-known logarithmic phase shift due to the tortoise radial coordinate, it is found that there is a mass-dependent logarithmic phase shift at infinity. This is found to be necessary to match the Newtonian cross-section in the non-relativistic limit. Finally, by imposing suitable boundary conditions near the horizon, the phase shift is calculated.

1 Introduction

The problem of a single scalar scattering off a Schwarzschild horizon, while admittedly academic, is perhaps a useful testing ground for techniques that might be applied elsewhere. As this case does not deal with the complication of considering spin, for the most part ignores possible interactions with surrounding matter, and ignores any lack of spherical symmetry in the horizon, this situation provides the opportunity to study how the particle is affected by the presence of a horizon, without other difficulties obstructing an understanding of the basic physical processes involved.

2 General features of the problem

The equation considered for the purpose of this paper is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) = m^2\psi \quad (1)$$

for a scalar with wave function ψ , mass m , in the presence of a background metric, written in Schwarzschild spherical coordinates as

$$g_{\mu\nu} = \begin{pmatrix} -(1+\Phi) & 0 & 0 & 0 \\ 0 & (1+\Phi)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (2)$$

where $\Phi = -2GM/r \equiv -R_S/r$, for M the mass of the black hole. Substituting the given metric, one easily sees that the wave equation becomes

$$\begin{aligned} -(1+\Phi)^{-1} \partial_t^2 \psi + (1+\Phi) \partial_r^2 \psi + r^{-2} \left[\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \psi) + \sin^{-2} \theta \partial_\phi^2 \psi \right] \\ + \frac{2+\Phi}{r} \partial_r \psi = m^2 \psi \end{aligned} \quad (3)$$

Inserting, as usual, the ansatz $\psi = \exp(-iEt) Y_{lm} R_l(r)$, this reduces to the radial equation

$$(1+\Phi)^2 \frac{d^2 R_l}{dr^2} + \frac{(2+\Phi)(1+\Phi)}{r} \frac{dR_l}{dr} + \left\{ E^2 - \left[m^2 + \frac{l(l+1)}{r^2} \right] (1+\Phi) \right\} R_l = 0 \quad (4)$$

Finally, changing variables to the “tortoise” coordinate $r^* = \int \frac{dr}{1+\Phi} = r + \ln \left| \frac{r}{R_S} - 1 \right|$ and $f_l(r^*) = r R_l(r)$ we find the “Schrodinger” form,

$$f_l'' + (E^2 - v_l) f_l = 0 \quad (5)$$

where

$$v_l \equiv m^2 - \frac{R_S m^2}{r} + \frac{l(l+1)}{r^2} - \frac{R_S [l(l+1) - 1]}{r^3} - \frac{R_S^2}{r^4} \quad (6)$$

Using the scaled radius $x = r/R_S$ and considering the dimensionless quantity v_l/m^2 , it becomes apparent that the mass only enters in the combination $(mR_S)^{-1}$, that is, the Compton wavelength of the particle in units of the Schwarzschild radius. Considering from this point forward only macroscopic black holes, it is clear that this parameter is extremely small, so that the angular momentum terms in v_l only become important for very large l . In this limit, the distinction between l^2 , $l(l+1)$, and $l(l+1) - 1$ can be ignored. Calling $(l/mR_S)^2 \equiv \alpha$, we finally have

$$v_l/m^2 = 1 - \frac{1}{x} + \frac{\alpha}{x^2} - \frac{\alpha}{x^3} \quad (7)$$

dropping the (very small) x^{-4} term. (See Figure 1.)

Written in this form, the general features of the problem are manifest. For $r \rightarrow \infty$, the tortoise coordinate approaches the original radial coordinate (to log accuracy) and the potential approaches a constant, so that we have

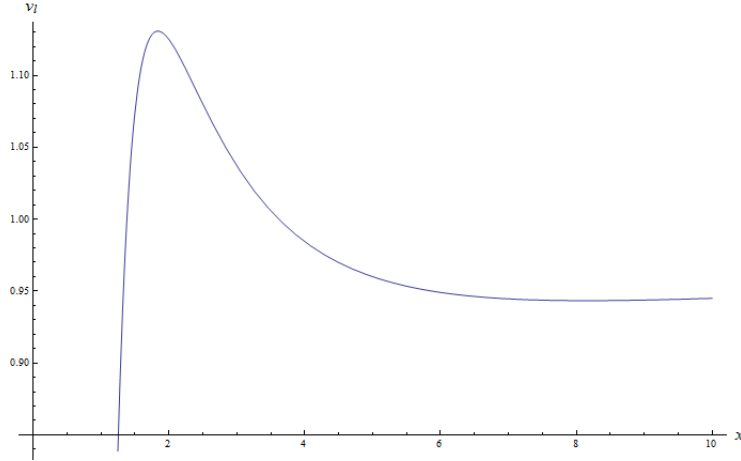


Figure 1: v_l/m^2 vs. x for $\alpha = 5$

$f_l \sim \exp(\pm ikr)$ as the dominant behavior, where $k^2 \equiv E^2 - m^2$, of course as expected. For $r^* \rightarrow -\infty$, i.e., $r \rightarrow R_S$, the potential goes to zero and we have $f_l(r^*) \exp(-iEt) \rightarrow \exp[-iE(t \pm r^*)]$, independent of the particle's mass.

Looking more closely at the large r limit, some complications arise. Although the dominant contribution to the phase grows linearly with distance, the log term in the tortoise radial coordinate introduces a phase shift which does not tend to a constant at infinity. This is also very much as expected, as the interaction range is infinite. We see that the magnitude of this phase shift at large distance is $kR_S \ln(r/R_S)$, as has been noted in the literature for some time [1]. However, even after changing the radial coordinate, we see that there is a Coulomb-type term in the potential, which should introduce an *additional* logarithmic phase shift at infinity, which has not been given sufficient attention thus far. Indeed, introducing an ansatz for the phase at infinity,

$$\phi(r) = kr + a \ln(r/R_S) \quad (8)$$

substituting straight-forwardly into the wave equation and collecting powers of $1/r$, we find that

$$a = kR_S \left(1 + \frac{m^2}{2k^2}\right) \quad (9)$$

The m -dependence does not violate the equivalence principle as one might at first think [2], because the scattering certainly may (and in fact, does) depend on the incoming particle's velocity relative to the black hole, which can be expressed in terms of m and k .

3 The Klein-Gordon equation in parabolic coordinates

3.1 Solution for forward scattering

Immediately, the question arises of what boundary conditions to impose at infinity. For a generic scattering problem, we impose on the wave function that, at large distance, it resemble a plane wave superimposed on an outgoing spherical wave. In this case, as with all long-range potentials, this is too much to ask for, as plane waves become logarithmically distorted at infinity. As is well known, Coulomb scattering becomes separable in parabolic coordinates, in which conditions at infinity are particularly simple to write. Although there is no hope of the wave equation in this case being completely separable, nevertheless this is a useful exercise, as scattering at large impact parameter, i.e., small scattering angle, will only probe the Coulomb tail of the potential, so the wave equation should be approximately separable in the forward scattering limit. We will see that this is in fact the case.

Introducing the coordinates $\eta = r + z$, $\xi = r - z$, it is straight-forward to see that

$$g_{\mu\nu} = \begin{pmatrix} -(1+\Phi) & 0 & 0 & 0 \\ 0 & \frac{1}{4} \left[\frac{\xi}{\eta} + (1+\Phi)^{-1} \right] & \frac{1}{4} \left[-1 + (1+\Phi)^{-1} \right] & 0 \\ 0 & \frac{1}{4} \left[-1 + (1+\Phi)^{-1} \right] & \frac{1}{4} \left[\frac{\eta}{\xi} + (1+\Phi)^{-1} \right] & 0 \\ 0 & 0 & 0 & \eta\xi \end{pmatrix} \quad (10)$$

where $\Phi = -2R_S/(\eta + \xi)$. Inverting this matrix, and substituting in (1) gives

$$\begin{aligned} \frac{4}{\xi + \eta} \left\{ \partial_\eta \left[\eta \left(1 - \frac{2R_S\eta}{(\xi + \eta)^2} \right) \partial_\eta \psi \right] + \partial_\xi \left[\xi \left(1 - \frac{2R_S\xi}{(\xi + \eta)^2} \right) \partial_\xi \psi \right] \right\} \\ = \left[m^2 - \frac{E^2}{1 + \Phi} \right] \psi \end{aligned} \quad (11)$$

For ordinary Coulomb scattering, the condition imposed on ψ is that it be equal to $\exp(ikz)f(\xi)$ for some function f that is independent of η [4]. It will be found that this condition is impossible to maintain for the case at hand. Instead, we can allow f to depend on both ξ and η , and examine the limit of large $r = (\xi + \eta)/2$ and small deflections in the hope that the resulting equations are then separable.

In this spirit, defining

$$f(\eta, \xi) = \exp[-ik(\eta - \xi)/2] \psi \quad (12)$$

and substituting in (11) we get, for any coordinate values,

$$\begin{aligned}
& -\frac{R_S}{2} \left\{ \frac{E^2}{1+\Phi} + k^2 \left(1 - \frac{2\xi\eta}{(\xi+\eta)^2} \right) \right\} f = \\
& \eta \left(1 + \Phi \frac{\eta}{\xi+\eta} \right) \partial_\eta^2 f + \left[1 + ik\eta \left(1 + \Phi \frac{\eta}{\eta+\xi} \right) + \Phi \frac{2\xi\eta}{(\xi+\eta)^2} \right] \partial_\eta f \\
& + \xi \left(1 + \Phi \frac{\xi}{\xi+\eta} \right) \partial_\xi^2 f + \left[1 - ik\xi \left(1 + \Phi \frac{\xi}{\eta+\xi} \right) + \Phi \frac{2\xi\eta}{(\xi+\eta)^2} \right] \partial_\xi f \quad (13)
\end{aligned}$$

This simplifies for the limits $r/R_S \gg 1$ and $\theta \rightarrow 0$. The first limit requires us to disregard Φ terms, the second to neglect $\frac{\xi\eta}{(\xi+\eta)^2}$, as this is just $\frac{\sin^2 \theta}{4}$. In this approximation, (13) becomes separable, so that we can finally require $f = f(\xi)$. The resulting equation is just

$$\xi f'' + (1 - ik\xi) f' + R_S k^2 \left(1 + \frac{m^2}{2k^2} \right) f = 0 \quad (14)$$

Comparing to the equation for a Coulomb potential [4], we see that this has an identical form, with the replacement $nk \rightarrow -R_S k^2 \left(1 + \frac{m^2}{2k^2} \right)$. Since the validity of this approximation rests on a large impact parameter, scattering in this limit should not be significantly affected by boundary conditions at the horizon. This means that we can deform the potential close to the horizon to a true Coulomb form without affecting the forward cross-section; borrowing the usual Rutherford result, we finally have, for small angles,

$$\frac{d\sigma}{d\Omega} = \frac{R_S^2}{4 \sin^4(\theta/2)} \left(1 + \frac{m^2}{2k^2} \right)^2 \rightarrow \frac{4R_S^2}{\theta^4} \left(1 + \frac{m^2}{2k^2} \right)^2 \quad (15)$$

Comparing with the result in Futterman *et al.* [3], we have precisely the mass dependence anticipated in (9).

3.2 Limiting cases

It is easy to see that mass dependence of this kind is necessary to approach the Newtonian scattering limit. Keeping in mind that n is just the potential strength divided by the velocity of the particle at large separation [4], in the Newtonian limit (small velocity, weak potential, and by extension small angle) the cross-section reduces to

$$\frac{d\sigma}{d\Omega} = \left(\frac{-mR_S/2}{2vk \sin^2(\theta/2)} \right)^2 = \frac{R_S^2}{4 \sin^4(\theta/2)} \left(\frac{m^2}{2k^2} \right)^2 \quad (16)$$

matching (15) for $m^2 \gg k^2$. In the ultra-relativistic limit, the cross section loses its momentum dependence, and becomes a constant $4R_S^2/\theta^4$. The meaning of this becomes clear when expressed in terms of impact parameter.

Since, by definition,

$$\frac{d\sigma}{d(\cos\theta)} d(\cos\theta) = 2\pi b \frac{db}{d\theta} d\theta \quad (17)$$

we can integrate (15) to find b . This gives, in the small angle limit, $\theta = 4GM/b$, corresponding exactly to the Einstein result for light-ray deflection.

4 The WKB technique

For larger deflections than those analyzed in the previous section, we can no longer set $\frac{\xi\eta}{(\xi+\eta)^2}$ to zero, and even for large radii the equation is not separable. Also the problem arises of setting boundary conditions at the horizon. The WKB approach is certainly a well-suited alternative for this problem. The phase in (8) certainly changes much more rapidly than the amplitude or frequency for any macroscopic black hole. However, the problem of boundary conditions at small $\frac{r}{R_S} - 1$ becomes still more pressing, as the phase integral is impossible to even define without them.

So far, the physical situation addressed has been unrealistic in a certain sense. One of the assumptions has been that there is absolutely no matter in the vicinity of the black hole other than the test particle in question. Of course, in any realistic case a black hole will be surrounded by a shell of accreted matter, which will interact with the scattered particle in an extremely complicated way. We can get past this obstruction by *defining* a black hole to be an object surrounded by an accretion layer so thin that an external particle is insensitive to the shell's internal dynamics.

Indeed, from (7) we see that the potential falls off exponentially as $r^* \rightarrow -\infty$, so that no l -dependence is felt from interaction with potential barriers at large negative r^* . Only an overall constant phase shift is left, which has no dynamical significance. For this restricted class of problems, we can therefore choose the model of interaction with surrounding matter for convenience. Choosing an infinite barrier at some fixed radius is especially simple, and corresponds to choosing a phase of zero at a fixed (small) radius. This will then be the lower limit of all phase integrals.

Referring back to (5) and (6), we see that corrections to the WKB approximation will be generally of order $(kR_S)^{-1}$ smaller than the dominant terms. By assumption, this is a very small parameter, and so can be neglected, except very near possible turning points. Then, keeping in mind the definition $\frac{dr^*}{dr} = \frac{1}{1+\Phi}$ and calling $a \sim R_S$ the cutoff radius, the phase of the wave-function is just

$$\phi(r) = kR_S \int_{a/R_S}^x dx \sqrt{1 + \frac{\beta}{x} - \frac{\gamma}{x^2} + \frac{\gamma}{x^3} \frac{x}{x-1}} \quad (18)$$

where $\beta \equiv \frac{m^2}{k^2}$ and $\gamma \equiv \alpha\beta = (l/kR_S)^2$. By expanding the integrand for large x we have yet another way of seeing the asymptotic result (9).

As we expect the dominant r and l dependence of the phase to be

$$\varphi(r) = kR_S \left[x - \frac{\pi}{2} \sqrt{\gamma} + \left(1 + \frac{\beta}{2} \right) \ln \left(\frac{x}{a/R_S} \right) \right] \quad (19)$$

[4], it is most interesting to look at the behavior of $\delta \equiv (\phi - \varphi)/kR_S$. For specific values of a/R_S and x , we can then plot the behavior of δ as a function of β and γ . (See figures 2 and 3.)

5 Conclusions

We see that the case of a massive scalar, as opposed to a massless one, does in fact modify the results quantitatively, while keeping all the basic qualitative features. This is not in violation of the equivalence principle, as the presence of the black hole defines a preferred reference frame, and the velocity of a particle in this frame is an important parameter. While it is not possible (or particularly desirable) to obtain a closed form expression for the cross-section at arbitrary angle, it is possible to find the forward cross-section for any value of the remaining parameters, which is the most physically relevant result for this problem. It has also been found that boundary conditions at the horizon can be dealt with simply, by treating accreted matter near the horizon to be an impenetrable barrier. This method is reasonable, as behavior at infinity is not affected by dynamics very near the horizon. This method of dealing with the horizon should be well suited for dealing with more complicated problems, e.g., spinor or vector scattering, which will be addressed in a future paper.

6 Acknowledgments

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References

- [1] J. A. H. Futterman, F.A. Handler, and R.A. Matzner, *Scattering from Black Holes* (Cambridge University Press, New York, 1988), p. 10.
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- [3] *ibid.*, p. 97.
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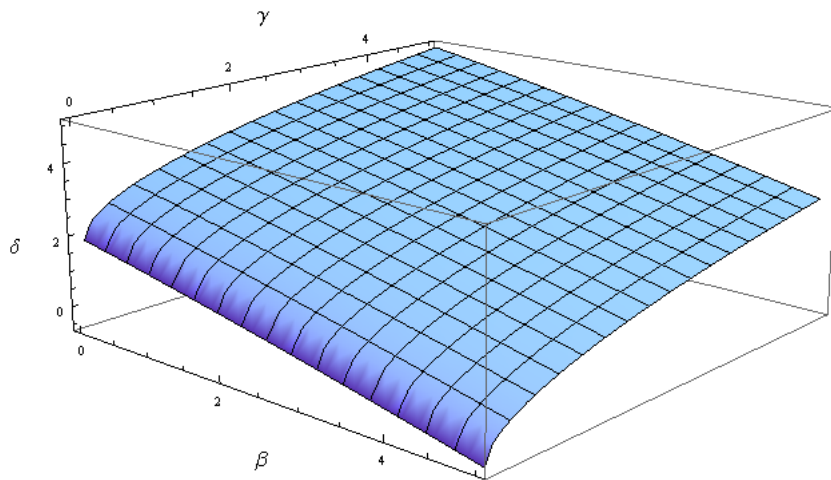


Figure 2: δ vs. β and γ for $x = 10$ and $a/R_S = 1.05$

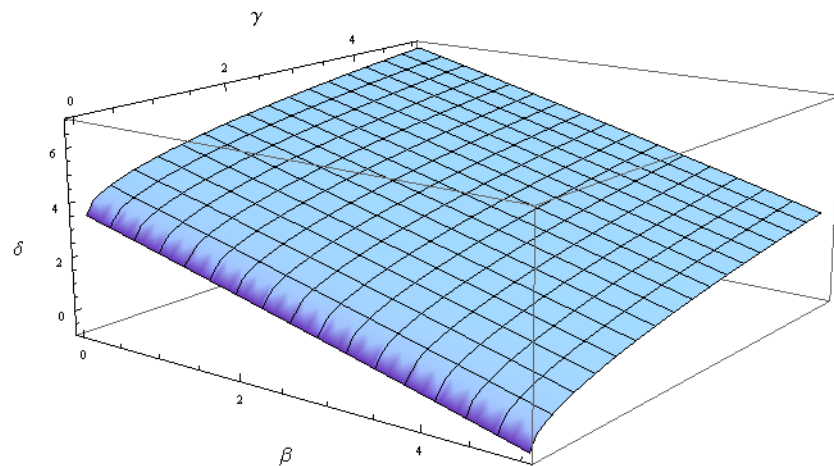


Figure 3: δ vs. β and γ for $x = 20$ and $a/R_S = 1.01$